3.3 Kodaira embedding theorem

Theorem 3.3.1 (Kodaira embedding theorem I '54). *A compact complex manifold M is projective if and only if M could be equipped with a positive line bundle.*

Let $\{s_i^p\}_{i=1}^{d_p}$ ($d_p := \dim H^0(M, L^p)$) be any orthonormal basis of $H^0(M, L^p)$ with respect to the usual L^2-norm (2.2.39). By Hodge theory, s_i^p's are holomorphic sections of L^p.

Definition 3.3.2. Let

$$\mathrm{Bl}_p := \{x \in M : s(x) = 0, \forall s \in H^0(M, L^p)\}, \tag{3.3.1}$$

which is called the base locus. The Kodaira map Φ_p is defined by

$$\Phi_p : M \backslash \mathrm{Bl}_p \to \mathbb{CP}^{d_p - 1}, \quad x \mapsto (s_1^p(x) : \cdots : s_{d_p}^p(x)). \tag{3.3.2}$$

Definition 3.3.3. Let L be a holomorphic line bundle.
It is called **semi-ample** if there exists p_0 such that for all $p \geq p_0$, $\mathrm{Bl}_p = \emptyset$.
It is called **ample** if it is semi-ample and Φ_p is an embedding.
It is called **very ample** if $\mathrm{Bl}_1 = \emptyset$ and Φ_1 is an embedding.

It is obvious that L is ample if and only if there exists p_0 such that for all $p \geq p_0$, L^p is very ample.

Theorem 3.3.4 (Kodaira embedding theorem II '54). *The holomorphic line bundle L is ample if and only if it is positive.*

For any $s \in H^0(M, L^p)$, we could write

$$s = \sum_{i=1}^{d_p} a_i s_i^p, \quad a_i \in \mathbb{C}. \tag{3.3.3}$$

Let γ be the tautological line bundle over $\mathbb{CP}^{d_p-1}$. For $([l], z) \in \gamma$, $z \in l \subset \mathbb{C}^{d_p}$. we define $\sigma_s \in \gamma^*$ such that for any $([l], z) \in \gamma$,

$$\langle \sigma_s([l]), ([l], z) \rangle := \sum_{i=1}^{d_p} a_i z_i. \tag{3.3.4}$$

Easy to see that for any $\zeta \in \gamma^*$, there exists $s \in H^0(M, L^p)$ such that $\zeta = \sigma_s$.

Proposition 3.3.5. *Let L be a semi-simple line bundle. Then for $p \geq p_0$, $\Phi_p : M \to \mathbb{CP}^{d_p - 1}$ is holomorphic and*

$$\Psi_p : \Phi_p^* \gamma^* \to L^p, \quad \Phi^* \sigma_s \mapsto s \tag{3.3.5}$$

defines a canonical isomorphism from $\Phi_p \gamma^*$ to L^p over M.

Proof. Since s_i^p's are holomorphic sections on L^p, from (3.3.2), Φ_p is holomorphic.

Let

$$S = (s_1^p(x), \cdots, s_{d_p}^p(x)) \in \mathbb{C}^{d_p}. \tag{3.3.6}$$

Then

$$\begin{aligned}\langle \Phi_p^* \sigma_s(x), \Phi_p^*(\Phi_p(x), S(x)) \rangle &= \langle \sigma(\Phi_p(x)), (\Phi_p(x), S(x)) \rangle \\ &= \sum_{i=1}^{d_p} a_i s_i^p(x) = s(x). \end{aligned} \tag{3.3.7}$$

Thus $\Phi_p^* \sigma_s(x) = 0$ if and only if $s(x) = 0$.

Since s and $\Phi_p^* \sigma_s$ are holomorphic sections of L^p and $\Phi_p^* \gamma^*$, Ψ_p is holomorphic. Thus it is continuous and the inverse of it is continuous.

The proof of this proposition is completed. □

Corollary 3.3.6. *If L is ample, then it is positive.*

Proof. If L is ample, then Φ_p is an embedding. Since γ^* is positive, $\Phi_p^* \gamma^*$ is positive. By Proposition 3.3.5, L^p is positive. So is L.

The proof of the corollary is completed. □

From now on, we assume that L is positive. From the Kodaira vanising theorem (Theorem 3.1.14),

$$H^q(M, L^p) = 0 \quad \text{for any } q \geq 0, p \gg 1. \tag{3.3.8}$$

Let P_p be the orthogonal projection from $\Omega^{0,*}(M, L^p)$ on to $\ker(\bar{\partial}^{L^p} + \bar{\partial}^{L^p,*}) = H^*(M, L^p)$. From the Kodaira vanishing theorem, if we only consider the case for p large enough,

$$P_p : \Omega^{0,*}(M, L^p) \to H^0(M, L^p). \tag{3.3.9}$$

Let

$$P_p(x, x') := \sum_{i=1}^{d_p} s_i^p(x) \otimes s_i^p(x)^* \in L_x^p \otimes (L^p)_{x'}^*. \tag{3.3.10}$$

Proposition 3.3.7. *For any* $s \in \Omega^{0,*}(M, L^p)$,

$$(P_p s)(x) = \int_{x' \in M} P_p(x, x') s(x') dx'. \tag{3.3.11}$$

Here $P_p(x, x')$ *is called the* ***Bergman kernel*** *associated with* L^p.

Proof. For any $s \in \Omega^{0,*}(M, L^p)$,

$$\begin{aligned}(P_p s)(x) &= \sum_{i=1}^{d_p} \left(\int_M s_i^p(x')^* \cdot s(x') dx' \right) \cdot s_i^p(x) \\ &= \int_M \left(\sum_{i=1}^{d_p} s_i^p(x) \otimes s_i^p(x')^* \right) \cdot s(x') dx' = \int_{x' \in M} P_p(x, x') s(x') dx'. \end{aligned} \tag{3.3.12}$$

The proof of this proposition is completed. □

Observe that $L^p \otimes (L^p)^*$ is a trivial line bundle. From (**??**), $P_p(x, x)$ is a complex valued function on M. If we take the adjoint with respect to h^{L^p}, we have

$$P_p(x, x) = \sum_{i=1}^{d_p} |s_i^p(x)|^2_{h^{L^p}}. \tag{3.3.13}$$

Proposition 3.3.8. *For any* $x \in M$,

$$h^{\Phi_p^* \gamma^*}(x) = P_p(x, x)^{-1} h^{L^p}(x). \tag{3.3.14}$$

Proof. Under the isomorphism (3.3.5), for any holomorphic section s on L^p, from (3.3.7) and (3.3.13),

$$\begin{aligned}\left|\Phi_p^* \sigma_s(x)\right|^2_{h^{\Phi_p^* \gamma^*}} = |\sigma_s(\Phi_p(x))|^2_{h^{\gamma^*}} &= \frac{|\langle \sigma_s(\Phi_p(x)), (\Phi_p(x), S(x)) \rangle|^2_{h^{L^p}}}{|(\Phi_p(x), S(x))|^2_{h^\gamma}} \\ &= \frac{|s(x)|^2_{L^p}}{\sum_{i=1}^{d_p} |s_i^p(x)|^2_{h^{L^p}}} = P_p(x, x)^{-1} |s(x)|^2_{L^p} \end{aligned} \tag{3.3.15}$$

The proof of this proposition is completed. □

The following theorem started from Tian '90 (also Bouche '90, Ruan '98) following the suggestion of Yau '87 was first established by Catlin '97, Zelditch '98.

Theorem 3.3.9. *For any $k, k' \in \mathbb{N}$, there exist $C_{k,k'} > 0$ and $b_r \in \mathscr{C}^\infty(M, \mathbb{C})$, $0 \le r \le k$ such that for any $p \in \mathbb{N}$,*

$$\left| P_p(x,x) - \sum_{r=0}^{k} b_r(x) p^{n-r} \right|_{\mathscr{C}^{k'}(M)} \le C_{k,k'} p^{n-k-1} \tag{3.3.16}$$

and

$$b_0 = \det\left(\frac{\dot{R}^L}{2\pi}\right). \tag{3.3.17}$$

Remark that Lu '00 and Lu-Tian '04 calculated b_1, b_2, b_3 used by Donaldson in his work on the existence of Kähler metrics with constant scalar curvature.

Proposition 3.3.10. *If L is positive, then it is semi-ample.*

Proof. If $R^L > 0$, by (3.1.50), $b_0 = \det\left(\dot{R}^L/2\pi\right) > 0$. From Theorem 3.3.9, for p large enough, $P_p(x,x) > 0$. Thus our proposition follows from (3.3.1) and (3.3.13).

The proof of our proposition is completed. □

Theorem 3.3.11 (Tian '90-Ruan '98)**.** *Assume that (L, h^L) is positive. Then the induced Fubini-Study metric $\frac{1}{p}\Phi_p^*(\omega_{FS})$ converges in $\mathscr{C}^\infty$-topology to $\omega = \sqrt{-1}R^L$. For any $l \ge 0$, there exists $C_l > 0$ such that*

$$\left| \frac{1}{p}\Phi_p^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l(M)} \le \frac{C_l}{p}. \tag{3.3.18}$$

Proof. From (1.2.34) and (2.1.46), we have

$$\omega_{FS} = \sqrt{-1}R^{\gamma^*} = \sqrt{-1}\bar{\partial}\partial \log |\sigma_s|^2_{h^{\gamma^*}}. \tag{3.3.19}$$

Thus from Proposition 3.3.8, (2.1.46) and (3.3.2),

$$\begin{aligned} \Phi_p^*\omega_{FS} &= \sqrt{-1}\bar{\partial}\partial \log |\Phi_p^*\sigma_s|^2_{h^{\gamma^*}} \\ &= \sqrt{-1}\bar{\partial}\partial \log |s(x)|^2_{h^{L^p}} - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \\ &= \sqrt{-1}R^{L^p} - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \\ &= p\omega - \sqrt{-1}\bar{\partial}\partial \log P_p(x,x) \end{aligned} \tag{3.3.20}$$

From Theorem 3.3.9, we have

$$\bar{\partial}\partial \log P_p(x,x) = \bar{\partial}\partial \log(p^n P_p(x,x)) = \bar{\partial}\partial \log b_0(x) + O(p^{-1}). \tag{3.3.21}$$

Thus our theorem follows directly from Theorem 3.3.9, (3.3.20) and (3.3.21). □

Proposition 3.3.12. *The Kodaira map $\Phi_p$is an immersion for $p \gg 1$.*

Proof. From Theorem 3.3.11, for any $l \geq 0$, there exists $C_l > 0$ such that

$$\left| \frac{1}{p} \Phi_p^*(g_{FS}) - g^{TM} \right|_{\mathscr{C}^l(M)} \leq \frac{C_l}{p}. \tag{3.3.22}$$

For $v \in T_xM$, $v \neq 0$, we have $g^{TM}(v, v) > 0$. From (3.3.22), for p large, we have $\Phi_p^*(g_{FS})(v, v) > 0$. It means that $g_{FS}(\Phi_* v, \Phi_* v) > 0$, which implies that $\Phi_* v \neq 0$.

The proof of our proposition is completed. □